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B.Sc. (Physics) Part-III, Paper-V, Group-A
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Tensor Part-III

Contravariant vector:

Tensors are defined using the properties of their transformation rules under coordinate transformations. Vectors are the special case of tensors.

Consider a physical enity is characterized by N functions A^i when expressed in the x^i coordinate system. Let the same enity be characterized by $\bar{A}^\alpha$ when it is measured in coordinate system $\bar{x}^\alpha$.

A^i are called components of a contravariant vector if they transform under coordinates transformations given as

$$\boxed{\bar{A}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^j} A^j} \quad \text{——} \quad (1)$$

which can be inverted to obtain A^i in terms of $\bar{A}^\alpha$ multiplying (1) by $\frac{\partial x^k}{\partial \bar{x}^\alpha}$ and summing over all α.

$$\frac{\partial x^k}{\partial \bar{x}^\alpha} \bar{A}^\alpha = \frac{\partial x^k}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^j} A^j$$

$$\frac{\partial x^k}{\partial \bar{x}^\alpha} \bar{A}^\alpha = \delta_j^k A^j = A^k$$

or

$$\boxed{A^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} \bar{A}^\alpha} \quad \text{——} \quad (2)$$

Covariant vector:

A set of N quantities A_i which are functions of the N coordinates x^i are said to be the components of a covariant vector if they transform given as

$$\bar{A}_\alpha = \frac{\partial x^j}{\partial \bar{x}^\alpha} A_j \quad \text{--- } (3)$$

under a change of coordinates from x^i to $\bar{x}^\alpha$, where $\bar{A}_\alpha$ are the components of the vector in the barred coordinate system. Inverse transformation is given by

$$A_i = \frac{\partial \bar{x}^\alpha}{\partial x^i} \bar{A}_\alpha \quad \text{--- } (4)$$

Q: Show that velocity and acceleration are contravariant vectors and that the gradient of a scalar field is a covariant vector.

Soln.

We have already obtain relations (see earlier class notes on tensor) given by

$$d\bar{x}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i \quad \text{--- } (5)$$

and $dx^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha$ — (8)

(i) Let t denotes time. Dividing Eq (8) by dt

$$\frac{\partial \bar{x}^{\alpha}}{dt} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{dx^{i}}{dt} \quad \text{—} (7)$$

Next, we define velocity components in barred and unbarred coordinate systems given by

$$\bar{v}^{\alpha} = \frac{d\bar{x}^{\alpha}}{dt} \quad , \quad v^{i} = \frac{dx^{i}}{dt}.$$

Now Eq. (7) can be written as

$$\boxed{\bar{v}^{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} v^{i}} \quad \text{—} (8)$$

See Eq. (1) and compare with (8). v^{i} is a contravariant vector. If you take derivative of (8) again we get,

$$\frac{d\bar{v}^{\alpha}}{dt} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{dv^{i}}{dt}$$

or

$$\boxed{\bar{a}^{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} a^{i}} \quad \text{—} (9) \quad \{ \; a \rightarrow \text{acceleration}$$

acceleration is also a contravariant vector.

(ii) Now consider $\phi \equiv \phi(x^{i})$ be a scalar field. Thus, its functions form will remain same under coordinate transformation

$$\phi(x^{i}) = \bar{\phi}(\bar{x}^{\alpha}) = \phi(\bar{x}^{\alpha})$$

Gradient of the scalar field will be a vector whose components can be defined by

$$A_i = \frac{\partial \phi}{\partial x^{i}} \quad , \quad \bar{A}_{\alpha} = \frac{\partial \bar{\phi}}{\partial \bar{x}^{\alpha}} = \frac{\partial \phi}{\partial \bar{x}^{\alpha}}$$

Next, using partial derivative, we can write

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^i}$$

or $$A_i = \frac{\partial \bar{x}^\alpha}{\partial x^i} \bar{A}_\alpha \quad \text{—— (10)}$$

See Eq. (4). and Eq. (10), It is clear that gradient of a scalar field is a covariant vector.

Tensor of Second Rank :

A set of N^2 functions A^{ij} are said to be the components of a contravariant tensor of rank two if they transform according to

$$\bar{A}^{\alpha\beta} = \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial \bar{x}^\beta}{\partial x^j} A^{ij} \quad \text{—— (11)}$$

under coordinate transformations where $\bar{A}^{\alpha\beta}$ are the components of the tensor in barred coordinate system.

Similarly for covariant tensor of second rank, we hav

$$\bar{A}_{\alpha\beta} = \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial x^j}{\partial \bar{x}^\beta} A_{ij}, \quad \text{—— (12)}$$

under the coordinate transformation

A set of N^2 functions A^i_j are said to be components of a tensor of contravariant rank one and covariant rank one (or mixed tensor of rank two) if they transform, under coordinate transformation, according to

$$\bar{A}^{\alpha}_{\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{\beta}} A^{i}_{j} \quad \text{—— (13)}$$

A_{ij} (the covariant tensor of rank two) can be represented by a square matrix

$$A_{ij} \equiv \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & & \ddots & \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix} \quad \text{—— (14)}$$

General form for a tensor of arbitrary rank:

$$\bar{A}^{\alpha_1, \alpha_2 \cdots \alpha_p}_{\beta_1, \beta_2 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} A^{i_1, i_2 \cdots i_p}_{l_1, l_2 \cdots l_q} \quad \text{(15)}$$

where $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ are a set of N^{p+q} functions and are said to be components of a tensor of contravariant rank p and covariant rank q. Total rank $(p+q)$.

and α_r (for $1 \le r \le p$), β_r (for $1 \le r \le q$) are free indices each having value between 1 and N, and i_r, l_s are dummy indices with summation over each from 1 to N.